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Generalized hexagons and polar spaces ☆

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Dedicated to the memory of Giuseppe Tallini

Abstract

Starting with the Tits' description of the Moufang hexagons, we discuss the construction of the known generalized hexagons as group coset geometries and some related topics. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

In [7], Kantor presented a construction of finite generalized polygons as group coset geometries, which was later investigated by Cohen and Cooperstein for even order (see [3]) and by the authors for odd order (see [2,10]). In the present paper, starting with the Tits' description of the Moufang hexagons given in [13], we discuss in detail the construction of the generalized hexagons associated with the algebraic groups ${}^3D_4(k)$ as group coset geometries and, for finite fields, we give a survey on the relations between the generalized hexagons and the polar spaces associated with a GF(q)-scroll of a Segre variety product of three lines.

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2. Generalized hexagons

A *generalized hexagon*, or simply a *hexagon*, is a bipartite graph with diameter 6 and girth 12.[1] These properties imply that a generalized hexagon is connected and every vertex has order at least 2. Here we only consider *thick* generalized hexagons, that is, we assume all vertices have order at least 3.

If Δ is a generalized hexagon and v is a fixed vertex of Δ, a rank 2 point–line geometry $H(\Delta,v)$ arises in the following way. The points of $H(\Delta,v)$ are the vertices of Δ at even distance from v, and the lines are the vertices of Δ at odd distance from v. A point and a line are incident if and only if they are connected via an edge of Δ. By construction, Δ is the incidence graph of $H(\Delta,v)$. The geometry $H(\Delta,v)$ is said to be a generalized hexagon as well.

As Δ is thick, both the number of points on a line and the number of lines through a point are constant. Moreover, if v and v' are two vertices at distance 1, then $H(\Delta,v')$ is the dual structure of $H(\Delta,v)$.

If Δ is finite, and the number of points on a line is $s+1$, the number of lines through a point is $t+1$, we say $H(\Delta,v)$ is generalized hexagon of order (s,t); clearly $H(\Delta,v')$ is a generalized hexagon of order (t,s).

Let Δ be a generalized hexagon, and let v_6 be a fixed vertex of Δ. Let $\Delta_i(v_6)$ be the set of vertices at distance i from v_6 for $i=1,2,3,4,5,6$, that is, a vertex v belongs to $\Delta_i(v_6)$ if and only if there is a minimal path $(v_6,v_5,\ldots,v_{6-i}=v)$ (here, if $6-i<j<6$, then v_j and v_{j+1} are distinct vertices of Δ connected by an edge, and $v_{j-1}\neq v_{j+1}$).

Put $\Delta_1(v_6)=\{v_j^{(5)}\mid j\in J\}$. Suppose there exists a group E of automorphisms of Δ which is simply transitive on the vertices at distance 6 from v_6, and fixes $\Delta_1(v_6)$ elementwise.

Let v_0 be a fixed vertex at distance 6 from v_6. For each j in J there is a unique path $(v_j^{(5)},v_j^{(4)},v_j^{(3)},v_j^{(2)},v_j^{(1)},v_0)$ of length 5 joining $v_j^{(5)}$ and v_0. Define

$$A_4(j)=\{g\in E\mid v_j^{(4)}g=v_j^{(4)}\},$$
$$A_3(j)=\{g\in E\mid v_j^{(3)}g=v_j^{(3)}\},$$
$$A_2(j)=\{g\in E\mid v_j^{(2)}g=v_j^{(2)}\},$$
$$A_1(j)=\{g\in E\mid v_j^{(1)}g=v_j^{(1)}\}.$$

We have $A_1(j)<A_2(j)<A_3(j)<A_4(j)$.

If w is an element of $\Delta_i(v_6)$ $(i=2,3,4,5)$, there is a unique path $\Gamma=(w_6=v_6,w_5,\ldots,w_{6-i}=w)$ of length i joining v_6 and w, which can be extended to some path $(w_6=v_6,w_5,\ldots,w_{6-i}=w,w_{6-(i+1)},\ldots,w_0)$ of length 6. If $w_5=v_j^{(5)}$, there is an element g of E such that $w_ig=v_j^{(i)}$ for $i=1,2,3,4,5$ and $w_0g=v_0$, because E is transitive on $\Delta_6(v_6)$, fixes $\Delta_1(v_6)$ elementwise, and $(v_j^{(5)},v_j^{(4)},v_j^{(3)},v_j^{(2)},v_j^{(1)},v_0)$ is the unique path of length 5 joining $v_j^{(5)}$ and v_0. So E is transitive on the set $\Delta_i(v_j^{(5)})$ for $i=1,2,3,4,5$.

[1] Here graphs are undirected, with no loops or double edges; a graph is bipartite if its cycles have even length.

Now define a new graph $\Delta(E, \{A_i(j) \mid i = 1,2,3,4;\ j \in J\})$ as follows. Vertices are: (i) distinguished vertices ∞ and $[j]$ for $j \in J$, (ii) right cosets in E of the subgroups $A_i(j)$ for $i = 1,2,3,4$ and $j \in J$, and (iii) elements of E. The vertex ∞ is adjacent precisely to all the vertices $[j]$, $j \in J$; the vertex $[j]$ is adjacent to ∞ and to all cosets of E over $A_4(j)$; for $i = 2,3,4$, $j \in J$ and $g,h \in E$ the vertex $A_i(j)g$ is adjacent to the vertex $A_{i-1}(j)h$ if it contains the latter; for each $j \in J$, the elements of E are adjacent to the cosets of $A_1(j)$ they belong to. It is easy to prove that the map θ from Δ to $\Delta(E, \{A_i(j) \mid i = 1,2,3,4;\ j \in J\})$, defined by

$$\begin{aligned}
&\theta: v_6 \mapsto \infty,\\
&\theta: v_j^{(5)} \mapsto [j],\\
&\theta: v_j^{(4)}g \mapsto A_4(j)g,\\
&\theta: v_j^{(3)}g \mapsto A_3(j)g,\\
&\theta: v_j^{(2)}g \mapsto A_2(j)g,\\
&\theta: v_j^{(1)}g \mapsto A_1(j)g,\\
&\theta: v_0 g \mapsto g,
\end{aligned}$$

is an isomorphism. The group E acts by right multiplication as an automorphism group on $\Delta(E, \{A_i(j) \mid i = 1,2,3,4;\ j \in J\})$.

The absence of cycles of length less the 12 in Δ is expressed by the condition:

(C) For $i_1, i_2, \ldots, i_h \in \{1,2,3,4\}$ such that $i_1 + i_2 + \cdots + i_h = 5$, and for $j_1, j_2, \ldots, j_h \in J$ such that $j_m \neq j_{m+1}$ for $1 < m < h-1$ and $j_1 \neq j_h$, we have $1 \notin A_{i_1}(j_1)^* A_{i_2}(j_2)^* \cdots A_{i_h}(j_h)^*$.[2]

If $H(\Delta, v_6)$ has finite order (s,t), then $J = \{1,2,\ldots,t+1\}$, the group E has order s^3t^2, and

(O) for all $j \in J$ we have $|A_1(u)| = s$, $|A_2(u)| = st$, $|A_3(u)| = s^2t$ and $|A_4(u)| = s^2t^2$.

In the finite case, it has been proven that conditions (C) and (O) are sufficient for the graph $\Delta(E, \{A_i(j) \mid i = 1,2,3,4;\ j \in J\})$ to be a generalized hexagon (see [3] or [7]).[3]

3. Moufang hexagons

Let Δ be a generalized hexagon and let $\Gamma = (v_0, v_1, v_2, v_3, v_4, v_5, v_6)$ be a path of length 6. By Theorem 4.1.1 of [12], the group $U(\Gamma)$ of all automorphisms of Δ fixing Γ and all the vertices adjacent to v_1, v_2, v_3, v_4, v_5 acts freely on the set of all cycles of length 12 containing Γ. The generalized hexagon is said to be *Moufang* if $U(\Gamma)$ is transitive (hence simply transitive) on that set for all paths Γ of length 6.

[2] As usual, H^* for a group H stands for $H\backslash\{1\}$.

[3] Notice that in Theorem 2.1 of [2] and of [10] we have implicitly assumed $A_3(j) = A_2(j)Z$, where Z is the commutator subgroup of E. As this is true for the group E used in those papers, the other results still hold.

Let us now choose a cycle of length 12 $(v_i \mid i \in \mathbf{Z}, v_{i+12} = v_i)$ in Δ and set, for all integers i, $U_i = U(v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5}, v_{i+6})$. If i and j are two integers such that $i < j$, denote by $U_{[i,j]}$ the subgroup of $\mathrm{Aut}\,\Delta$ generated by all U_r with $i \leqslant r \leqslant j$. As usual, put $[x, y] = xyx^{-1}y^{-1}$ and let $[X, Y]$ be the group generated by all $[x, y]$ for $x \in X$ and $y \in Y$, for X, Y any two subgroups of $\mathrm{Aut}\,\Delta$. It is well known (cf. [13]) that if Δ is a Moufang hexagon the groups U_i have the following properties:

(1) $U_i = U_{i+12} \neq 1$ for all i;

(2) $[U_i, U_j] \subset U_{[i+1,j-1]}$ for $i < j < i+6$;

(3) for any integer i and any u in U_i^*, there is an element m in $U_{i+6}uU_{i+6}$ such that, for all integers j, the group ${}^mU_j$ (conjugate of U_j by m) is equal to $U_{2i+12-j}$;

(4) if U_+ denotes the group generated by U_1, U_2, U_3, U_4, U_5, U_6, the product map $U_1 \times U_2 \times U_3 \times U_4 \times U_5 \times U_6 \mapsto U_+$ is bijective.

Lemma 1. *The group $E = U_1U_2U_3U_4U_5$ is simply transitive on the vertices of Δ at distance 6 from v_6, and $Q = U_2U_3U_4U_5U_6$ is simply transitive on the vertices of Δ at distance 6 from v_7.*

Proof. Let w be a vertex at distance 6 from v_6. Then $d(w, v_5) = 5$. Denote by $(v_5, w_4, w_3, w_2, w_1, w)$ the unique path of length 5 joining v_5 and w. We have $w_4 \neq v_6$, $w_3 \neq v_5$, $w_2 \neq v_4$, $w_1 \neq v_3$ and $w \neq v_2$, otherwise $d(w, v_5) < 5$. As U_i is simply transitive on the vertices at distance 1 from v_i different from v_{i+1}, there exist and are uniquely defined elements $g_i \in U_i$ for $i = 1, 2, 3, 4, 5$ such that $w_4g_5 = v_4$, $w_3g_5g_4 = v_3$, $w_2g_5g_4g_3 = v_2$, $w_1g_5g_4g_3g_2 = v_1$, and $wg_5g_4g_3g_2g_1 = v_0$. This implies that E is simply transitive on $\Delta_6(v_6)$. With similar arguments, one can prove Q is simply transitive on $\Delta_6(v_7)$. $\square$

The enumeration of all Moufang hexagons is given by Tits in [13] (see, also, Faulkner in [4]). Here we are interested in the generalized hexagon associated with the algebraic group ${}^3D_4(k)$. The required information is on pp. 244–245 of [13]. However, we are making this somewhat more explicit in order to facilitate later calculations.

Let K be a Galois extension of degree 3 of k, and denote by τ a generator of the Galois group of K over k, that is, $\tau^3 = 1$ and τ is the identity over k. Let $N(x) = x^{1+\tau+\tau^2}$, and $\mathrm{Tr}(x) = x + x^\tau + x^{\tau^2}$.

If $U_+ = k \times K \times k \times K \times k \times K$, denote by x_i the element with ith coordinate equal to x and all others 0, and let U_i denote the set of all such elements x_i. Then U_1, U_3, U_5 (resp. U_2, U_4, U_6) are isomorphic to $k(+)$ (resp. to $K(+)$) via $x \mapsto x_i$. The commutator relations

$$[x_1, y_6] = (0, xy, x^2N(y), -xy^{\tau+\tau^2}, -xN(y), 0),$$

$$[x_1, y_5] = (0, 0, xy, 0, 0, 0),$$

$$[x_2, y_6] = (0, 0, \mathrm{Tr}(x^{\tau+\tau^2}y), x^\tau y^{\tau^2} + y^\tau x^{\tau^2}, \mathrm{Tr}(y^{\tau+\tau^2}x), 0),$$

$$[x_2, y_4] = (0, 0, \mathrm{Tr}(xy), 0, 0, 0),$$

$$[x_4, y_6] = (0, 0, 0, 0, -\mathrm{Tr}(xy), 0),$$

and $[U_i, U_j] = 1$ for all $(i, j) \notin \{(1,6),(1,5),(2,6),(2,4),(4,6)\}$ determine a group U_+. We can define a generalized hexagon $\Delta(U_+)$ by using U_+ and a closed cycle $\Sigma = (v_n \mid n \in \mathbf{Z}, v_{n+12} = v_n)$ of length 12. Consider the graph $\bar{\Delta} = U_+ \times \Sigma$ consisting of disjoint copies of Σ indexed by U_+. Introduce the following equivalence relation $\sim$ in the set of vertices of $\bar{\Delta}$: $(u, v_i) \sim (u', v_j)$ if and only if $i = j$ and either $u^{-1}u' \in U_{[1,i]}$ if $1 \leqslant i \leqslant 6$ or $u^{-1}u' \in U_{[i-6,6]}$ if $7 \leqslant i \leqslant 12$. Then $\Delta(U_+) = \bar{\Delta}/\sim$, that is, the vertices of Δ are the equivalence classes of vertices of $\bar{\Delta}$ and its edges are the images of the edges of $\bar{\Delta}$ under the canonical map $\bar{\Delta} \mapsto \bar{\Delta}/\sim$. One can prove that $\Delta(U_+)$ is the Moufang hexagon associated with the algebraic group ${}^3D_4(k)$ (cf. [13]).

For $i = 2, 4, 6$ let $\bar{U}_i = \{x_i \mid x \in k\}$. Then $\bar{U}_+ = U_1 \bar{U}_2 U_3 \bar{U}_4 U_5 \bar{U}_6$ is a group and $\Delta(\bar{U}_+)$ is the generalized hexagon associated with the algebraic group $G_2(k)$ (cf. [13]).

We can prove directly that $E = U_1 U_2 U_3 U_4 U_5$ is isomorphic to the group $k \times K \times k \times K \times k$ with the operation

$$(\alpha, b, \gamma, d, \varepsilon)(\alpha', b', \gamma', d', \varepsilon') = (\alpha + \alpha', b + b', \gamma + \gamma' + \alpha'\varepsilon - \mathrm{Tr}(b'd), d + d', \varepsilon + \varepsilon').$$

The center is $Z = \{(0, 0, \gamma, 0, 0) \mid \gamma \in k\}$ which is also the commutator subgroup of E. We remark that E/Z is an 8-dimensional k-vector space equipped with a bilinear form defined by $\boldsymbol{b}(gZ, hZ) = [g, h]$. If k has characteristic different from 2, then the bilinear form is a non-singular alternating form. When k has characteristic 2, the bilinear form is associated with the quadratic map $gZ \mapsto g^2$.

Moreover $\bar{E} = E \cap \bar{U}_+ = U_1 \bar{U}_2 U_3 \bar{U}_4 U_5$ is a subgroup of E such that $\bar{E}/Z$ is a 4-dimensional non-singular k-subspace of E/Z.

Let $\tilde{K} = K \cup \{\infty\}$. The group U_6 is transitive on the vertices of $\Delta_1(v_6)$ different from v_7. If t_6 is an element of U_6, let $v_t^{(5)}$ be the image of v_5 under t_6, and let $v_\infty^{(5)} = v_7$. Then $\Delta_1(v_6) = \{v_t^{(5)} \mid t \in \tilde{K}\}$. In order to use notations of Section 2, put

$$(v_\infty^{(5)}, v_\infty^{(4)}, v_\infty^{(3)}, v_\infty^{(2)}, v_\infty^{(1)}) = (v_7, v_8, v_9, v_{10}, v_{11}),$$

$$(v_0^{(5)}, v_0^{(4)}, v_0^{(3)}, v_0^{(2)}, v_0^{(1)}) = (v_5, v_4, v_3, v_2, v_1).$$

Therefore

$$A_4(\infty) = U_2 U_3 U_4 U_5, \quad A_3(\infty) = U_3 U_4 U_5, \quad A_2(\infty) = U_4 U_5, \quad A_1(\infty) = U_5,$$
$$A_4(0) = U_1 U_2 U_3 U_4, \quad A_3(0) = U_1 U_2 U_3, \quad A_2(0) = U_1 U_2, \quad A_1(0) = U_1.$$

As U_6 normalizes E, we have $A_i(t) = t_6^{-1} A_i(0) t_6$ for all t in K and for $i = 1, 2, 3, 4$. As for all t in K we have

$$\begin{aligned} t_6^{-1}(\alpha, b, \gamma, d, \varepsilon) t_6 = (&\alpha, b + \alpha t, \gamma - \alpha^2 t^{1+\tau+\tau^2} - \mathrm{Tr}(b^{\tau+\tau^2} t) - \mathrm{Tr}(\alpha b t^{\tau+\tau^2}), \\ &d + \alpha t^{\tau+\tau^2} + {}^\tau t^{\tau^2} + b^{\tau^2} t^\tau, \varepsilon + \alpha t^{1+\tau+\tau^2} + \mathrm{Tr}(b t^{\tau+\tau^2}) + \mathrm{Tr}(dt)), \end{aligned}$$

by a direct calculation we obtain

$$\begin{aligned} A_4(t) &= \{(\alpha, b, \gamma, d, \alpha t^{1+\tau+\tau^2} - \mathrm{Tr}(b t^{\tau+\tau^2} - dt)) \colon \alpha, \gamma \in k; b, d \in K\}, \\ A_3(t) &= \{(\alpha, b, \gamma, -\alpha t^{\tau+\tau^2} + b^\tau t^{\tau^2} + b^{\tau^2} t^\tau, -2\alpha t^{1+\tau+\tau^2} + \mathrm{Tr}(b t^{\tau+\tau^2})) \colon \\ &\qquad \alpha, \gamma \in k; b \in K\}, \end{aligned}$$

$$A_2(t) = \{(\alpha, b, -\alpha^2 t^{1+\tau+\tau^2} - \mathrm{Tr}(b^{\tau+\tau^2} t - \alpha b t^{\tau+\tau^2}), -\alpha t^{\tau+\tau^2} + b^\tau t^{\tau^2} + b^{\tau^2} t^\tau, -2\alpha t^{1+\tau+\tau^2} + \mathrm{Tr}(b t^{\tau+\tau^2})): \alpha \in k; b \in K\},$$

$$A_1(t) = \{(\alpha, \alpha t, -\alpha^2 t^{1+\tau+\tau^2}, \alpha t^{\tau+\tau^2}, \alpha t^{1+\tau+\tau^2}): \alpha \in k\}.$$

We notice that $A_3(t) = A_2(t)Z$. By construction

$$\Delta(E, \{A_i(t) \mid i = 1,2,3,4; t \in \tilde{K}\})$$

is isomorphic to the ${}^3D_4(k)$-hexagon $\Delta(U_+)$ and

$$\Delta(\bar{E}, \{A_i(t) \cap \bar{E} \mid i = 1,2,3,4; t \in \tilde{k} = k \cup \{\infty\}\})$$

is isomorphic to the $G_2(k)$-hexagon $\Delta(\bar{U}_+)$.

4. The finite case

Let V be a vector space over the field k. Denote by $\mathrm{PG}(V,k)$ the projective space associated with V. For A a vector subspace of V, denote by $P(A)$ the projective subspace of $\mathrm{PG}(V,k)$ associated with A. If $A = \langle v \rangle$ has dimension 1, $P(v)$ is the point of $\mathrm{PG}(V,k)$ associated with A. If V has finite dimension $n+1$ over the field $k = \mathrm{GF}(q)$, write $\mathrm{PG}(n,q)$ instead of $\mathrm{PG}(V,k)$.

Let Σ^* be a projective space. A subset Σ of points of Σ^* is a *subgeometry* of Σ^* if there is a set $\mathscr{L}$ of subsets of Σ with the following properties:

(1) each element of $\mathscr{L}$ is contained in a line of Σ^*;
(2) $(\Sigma, \mathscr{L})$ is a projective space;
(3) if a line l of Σ^* contains two points of Σ, then $l \cap \Sigma \in \mathscr{L}$;
(4) no line of Σ^* belongs to $\mathscr{L}$.

Let Σ be a subgeometry of $\Sigma^* = \mathrm{PG}(n, q^r)$, and suppose Σ is isomorphic to $\mathrm{PG}(m,q)$. We say that Σ is a *canonical* subgeometry of Σ^* if $\Sigma^* = \langle \Sigma \rangle$ and $n = m$. If this is the case, and $\Sigma = \mathrm{PG}(V,k)$, $\Sigma^* = \mathrm{PG}(V^*, K)$, a basis of V over $k = \mathrm{GF}(q)$ is also a basis of V^* over $K = \mathrm{GF}(q^r)$, i.e. $V^* = K \otimes V$.

Let Σ be a canonical subgeometry of Σ^*. For each subspace S of Σ^*, the set $S \cap \Sigma$ is a subspace of Σ. A subspace S of Σ^* is said to be a subspace of Σ whenever S and $S \cap \Sigma$ have the same dimension.

From now on, let $K = \mathrm{GF}(q^3)$, $k = \mathrm{GF}(q)$, and let V_1, V_2, V_3 be three vector spaces of dimension 2 over K. The vector space $V^* = V_1 \otimes V_2 \otimes V_3$ has dimension 8 over K and $\mathrm{PG}(V^*, K) \cong \mathrm{PG}(7, q^3)$. Let

$$S_{2,2,2}(q^3) = \{P(v_1 \otimes v_2 \otimes v_3) \in \mathrm{PG}(V^*, K): v_i \in V_i,\ i = 1,2,3\}.$$

Thus, $S_{2,2,2}(q^3)$ is a Segre variety in $\mathrm{PG}(7, q^3)$ of type $(2,2,2)$. Define

$$\mathscr{R}_1 = \{P(V_1 \otimes Kv_2 \otimes Kv_3): v_2 \in V_2, v_3 \in V_3\},$$
$$\mathscr{R}_2 = \{P(Kv_1 \otimes V_2 \otimes Kv_3): v_1 \in V_1, v_3 \in V_3\},$$
$$\mathscr{R}_3 = \{P(Kv_1 \otimes Kv_2 \otimes V_3): v_1 \in V_1, v_2 \in V_2\}.$$

The elements of $\mathcal{R}_i$ $(i=1,2,3)$ are lines of $\mathrm{PG}(7,q^3)$ contained in $S_{2,2,2}(q^3)$. Moreover, two lines of the same $\mathcal{R}_i$ are disjoint and any point of $S_{2,2,2}(q^3)$ belongs to a line of each $\mathcal{R}_i$. If $p = P(v_1 \otimes v_2 \otimes v_3)$ is a point of $S_{2,2,2}(q^3)$, let r_i be the line of $\mathcal{R}_i$ incident with p $(i = 1,2,3)$. The 3-dimensional subspace $T_p^* = \langle r_1, r_2, r_3\rangle$ is the tangent space of $S_{2,2,2}(q^3)$ at the point p. We can prove by a direct calculation that $T_p^* \cap S_{2,2,2}(q^3) = r_1 \cup r_2 \cup r_3$.

If e_1, e_2 is a basis of V_1, e_3, e_4 is a basis of V_2 and e_5, e_6 is a basis of V_3, put $v_1 = e_1 \otimes e_3 \otimes e_5$, $v_2 = e_1 \otimes e_3 \otimes e_6$, $v_3 = e_1 \otimes e_4 \otimes e_5$, $v_4 = e_1 \otimes e_4 \otimes e_6$, $v_5 = e_2 \otimes e_3 \otimes e_5$, $v_6 = e_2 \otimes e_3 \otimes e_6$, $v_7 = e_2 \otimes e_4 \otimes e_5$, $v_8 = e_2 \otimes e_4 \otimes e_6$, and denote by $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ the homogeneous coordinates of a point of $\mathrm{PG}(7,q^3)$ with respect to this basis of V^*.

Let σ be the semilinear collineation of $\mathrm{PG}(7,q^3)$ defined by

$$(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)^\sigma = (x_1^q, x_3^q, x_5^q, x_7^q, x_2^q, x_4^q, x_6^q, x_8^q).$$

The points fixed by σ are exactly those belonging to the canonical subgeometry $\mathrm{PG}(7,q) = \{(\alpha, c^{q^2}, c^q, b, c, b^q, b^{q^2}, \delta) \mid \alpha, \delta \in k; b, c \in K\}$ of $\mathrm{PG}(7,q^3)$. The set $\mathcal{O}_3 = S_{2,2,2}(q^3) \cap \mathrm{PG}(7,q)$ is called the k-scroll of $S_{2,2,2}(q^3)$.

A point p of $\mathrm{PG}(7,q)$ belongs to $\mathcal{O}_3$ if and only if either $p = P(v_1)$ or there is an element $a \in K$ such that $p = (a^{1+q+q^2}, a^{1+q}, a^{1+q^2}, a, a^{q+q^2}, a^q, a^{q^2}, 1)$ (see [9] or [10]).

Let E be the group constructed in Section 3 and let $V = E/Z$ be the 8-dimensional vector space defined by E. We can suppose $V \cong k \times K \times K \times k$, so that the elements of V are of type (α, b, c, δ) with $\alpha, \delta \in k$ and $b, c \in K$. The bilinear form $\boldsymbol{b}$ of V is defined by $\boldsymbol{b}((\alpha, b, c, \delta), (\alpha', b', c', \delta')) = \alpha\delta' - \delta\alpha' + \mathrm{Tr}(bc' - b'c)$. If $\mathrm{char}(k) = 2$, the quadratic form $gZ \mapsto g^2$ is $(\alpha, b, c, \delta) \mapsto \alpha\delta + \mathrm{Tr}(bc)$.

If $p_t = A_1(t)Z/Z$, and $T_t = A_3(t)/Z$, then $\mathcal{O}_3 = \{p_t \mid t \in \tilde{K}\}$ is the k-scroll of a Segre variety $S_{2,2,2}(q^3)$ and T_t is the tangent space of $\mathcal{O}_3$ at p_t. For all $t \in \tilde{K}$, T_t is a maximal totally isotropic subspace with respect to the polarity defined by $\boldsymbol{b}$, and $\mathcal{S}_3 = \{T_t \mid t \in \tilde{K}\}$ is a partial spread of $\mathrm{PG}(7,q)$. It is easy to see that $A_4(t)/Z$ is the polar hyperplane of p_t with respect to the polarity defined by $\boldsymbol{b}$. Moreover $\mathcal{C} = \{p_t \mid t \in \tilde{k}\}$ is a twisted cubic; it is the intersection of $\mathcal{O}_3$ with the 3-dimensional subspace $S = P(k \times k \times k \times k)$ of $\mathrm{PG}(7,q)$ (see [8,9] and [10], Section 3).

If k has even characteristic, let $Q^+(7,q)$ be the hyperbolic quadric with equation $\alpha\delta + \mathrm{Tr}(bc) = 0$. An *ovoid* of $Q^+(7,q)$ is a set of point which has exactly one point in common with each maximal totally singular subspace of $Q^+(7,q)$. A *spread* of $Q^+(7,q)$ is a set of $q^3 + 1$ pairwise disjoint totally singular subspaces of $Q^+(7,q)$, for more details, see [11]. Then $\mathcal{O}_3$ is an ovoid and $\mathcal{S}_3$ is a spread of the polar space $Q^+(7,q)$ defined by the quadratic form $\bar{g} \mapsto g^2$ (see [5], [6] and [9]). The subgroups $A_3(t)$ and $A_4(t)$ of E such that $\mathcal{S}_3 = \{A_3(t)/Z \mid t \in \tilde{K}\}$ and $A_4(t)/Z = p_t^\perp$ are uniquely defined. But, as q is even, the subgroups $A_1(t)$ and $A_2(t)$ such that $A_1(t)Z/Z = p_t$ and $A_3(t) = A_2(t)Z$ are not uniquely defined. Thus, the construction of a generalized hexagon from the same group E with possibly distinct ovoids and spreads impinges on the existence of a lift from V to E such that condition (C) holds for the following subscript sequences (as proven in [3]):

(a) $h=3$ and $i_1,i_2,i_3\in\{1,2\}$;

(b) $h=4$ and $i_1,i_2,i_3,i_4\in\{1,2\}$;

(c) $h=5$ and $i_1=i_2=i_3=i_4=i_5=1$.

If k has odd characteristic, let $W(7,q)$ be the symplectic space associated with the polarity $\perp$ defined by the alternating form $\boldsymbol{b}$. Then $\mathcal{O}_3$ is a partial ovoid and $\mathcal{S}_3$ is a partial spread [4] of $W(7,q)$ of size q^3+1 such that each point of $\mathcal{O}_3$ belongs to an element of $\mathcal{S}_3$ (see [10]). Let p_t be a fixed point of $\mathcal{O}_3$, and let $p_t^{\perp}$ be the polar hyperplane of p_t with respect to $\perp$. If v is an element of $\tilde{K}$ and $v\neq t$, then $p_t^{\perp}\cap p_v^{\perp}=W(5,q)$ because p_t and p_v are not conjugate in $W(7,q)$. For all $u\in\tilde{K}$, let $\alpha_u=T_u\cap p_t^{\perp}$. If $t\neq u$, then α_u is a plane, otherwise $\alpha_t=T_t$. If $X_u=\langle p_t,\alpha_u\rangle$, then X_u is a maximal totally isotropic subspace of $W(7,q)$ and $\beta_u=X_u\cap p_t^{\perp}\cap p_v^{\perp}$ is a plane of $W(5,q)$. Then $\mathcal{F}_p=\{\beta_u\mid u\in\tilde{K}\}$ is a symplectic spread of $W(5,q)$, associated with a twisted field of dimension three over its center (see [1]).

Let $W(2n+1,q)$ be the polar space arising from the symplectic polarity of PG$(2n+1,q)$ associated with the non-singular alternating bilinear form (,) of V. Let us denote by $\mathcal{O}$ and $\mathcal{P}$ respectively a partial ovoid and a partial spread of $W(2n+1,q)$ such that $|\mathcal{O}|=|\mathcal{P}|=q^n+1$ and such that for each $p\in\mathcal{O}$ there exists a unique $T_p\in\mathcal{P}$ such that $p\in T_p$. If $p_i=P(v_i)$ $(i=1,2,3,4)$ are distinct points of $\mathcal{O}$, put

$$Q(p_1,p_2,p_3,p_4)=\left\{P\left(\sum\nolimits_{i=1}^{4}x_iv_i\right):\ \sum_{i,j=1,i\neq j}^{4}x_ix_j(v_i,v_j)=0\right\},$$

$$C(p_1,p_2,p_3)=Q(p_1,p_2,p_3,p_4)\cap\langle p_1,p_2,p_3\rangle.$$

Suppose that, for all pairwise distinct p_1,p_2,p_3,p_4 in $\mathcal{O}$:

(a) $\langle T_{p_1},p_2\rangle\cap\mathcal{O}=\{p_1,p_2\}$;

(b) no plane contains four points of $\mathcal{O}$;

(c) any line of $W(2n+1,q)$ joining a point of T_{p_1} with a point of T_{p_2} is not incident with p_3;

(d) if $r\in\langle p_1,p_2,p_3\rangle\cap T_{p_4}$, then r does not belong to $C(p_1,p_2,p_3)$;

(e) no five points of $\mathcal{O}$ belong to $Q(p_1,p_2,p_3,p_4)$.

Let $W(2n+3,q)$ be the polar space arising from a symplectic polarity $\perp$ of PG$(2n+3,q)$. If x and y are two points of PG$(2n+3,q)$ not collinear in $W(2n+3,q)$, then $x^{\perp}\cap y^{\perp}=T$ is a $(2n+1)$-dimensional subspace of PG$(2n+3,q)$ and $T\cap W(2n+3,q)\cong W(2n+1,q)$. Define a point-line geometry $H(\mathcal{O},\mathcal{P})$ in the following way:

Points:

(1) the distinguished point x;

(2) the points of PG$(2n+3,q)$ different from x but contained in one of the lines $\langle x,p\rangle$ where p belongs to $\mathcal{O}$;

[4] A *partial ovoid* of $W(2n+1,q)$ is a set of pairwise non-orthogonal points. A *partial spread* of $W(2n+1,q)$ is a set of pairwise disjoint maximal totally isotropic subspaces.

(3) the maximal totally isotropic subspaces of $W(2n+3,q)$ not contained in $x^{\perp}$ and meeting one of the $(n+1)$-dimensional subspaces $\langle x, T_p\rangle$ $(p\in\mathcal{O})$ in a n-dimensional subspace;

(4) the points of $\mathrm{PG}(2n+3,q)\backslash x^{\perp}$.

Lines:

(i) the lines $\langle x, p\rangle$ where $p\in\mathcal{O}$;

(ii) the n-dimensional subspaces not incident with x, and contained in one of the $(n+1)$-dimensional subspaces $\langle x, T_p\rangle$ $(p\in\mathcal{O})$;

(iii) the totally singular lines not contained in $x^{\perp}$ and meeting $x^{\perp}$ in a point belonging to one of the lines $\langle x, p\rangle$ $(p\in\mathcal{O})$.

Incidences:

Points of type (2) and lines of type (iii) are never incident. All other incidences are inherited from $\mathrm{PG}(2n+3,q)$.

In [10] it has been proven that

Theorem 1. *If q is odd, $H(\mathcal{O},\mathcal{P})$ is a generalized hexagon with parameters (q,q^n). Moreover $H(\mathcal{O}_3,\mathcal{S}_3)$ is isomorphic to the ${}^3D_4(q)$-hexagon.*

Let S be a 3-dimensional subspace of T such that $S\cap\mathcal{O}_3$ is a twisted cubic. If p_a is a point of $\mathcal{O}_3$, then $T_{p_a}\cap S$ is a line if and only if $p_a\in S$. Let $U=\langle x,S,y\rangle=\mathrm{PG}(5,q)$. Define a point-line geometry $H(S)$ in the following way. A point z of $H(\mathcal{O}_3,\mathcal{S}_3)$ of type (1) or (2) or (4) is a point of $H(S)$ if and only if z is a point of U. A point X of $H(\mathcal{O}_3,\mathcal{S}_3)$ of type (3) is a point of $H(S)$ if and only if $U\cap X$ is a plane. A line l of $H(\mathcal{O}_3,\mathcal{S}_3)$ of type either (i) or (iii) is a line of $H(S)$ if and only l is contained in U. A line X of $H(\mathcal{O}_3,\mathcal{S}_3)$ of type (ii) is a line of $H(S)$ if and only if $X\cap U$ is a line. Therefore, $H(S)$ can be regarded as a substructure of $H(\mathcal{O}_3,\mathcal{S}_3)$. In particular, each chain in $H(S)$ defines a chain of $H(\mathcal{O}_3,\mathcal{S}_3)$ of the same length.

Theorem 2 (Bader and Lunardon [2] and Lunardon [10]). *If q is odd, $H(S)$ is a generalized hexagon isomorphic to the $G_2(q)$-hexagon.*

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